Proof of Orthogonality Relations part 1

$\varphi : G \to GL(V)$ irreducable.

$\chi := \chi_\varphi$

Show: $\langle \chi, \chi \rangle = 1$

WLOG: $\varphi : G \to GL_n(\mathbb{C})$, $\varphi_g = (\varphi_{ij}(g))$

Recall: $\chi(g^{-1}) = \overline{\chi(g)}$

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\, \chi(g^{-1})$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{i,j} \varphi_{ii}(g)\, \varphi_{jj}(g^{-1})$$

$1 \leq i,j \leq n$

$$= \sum_{i,j} \underbrace{\frac{1}{|G|} \sum_{g \in G} \varphi_{ii}(g)\, \varphi_{jj}(g^{-1})}_{\frac{1}{n}\delta_{ij}}$$

Lemma: $\varphi: G \to GL(V)$ irreducible, $\dim V = n$. $T \in \mathrm{Hom}(V,V)$. Define

$$T' := \frac{1}{|G|} \sum_{g \in G} \varphi_g T \varphi_{g^{-1}}.$$

Then $T' = \lambda I$, where $\lambda = \frac{1}{n} \mathrm{Tr}(T)$

Proof: $T' \in \mathrm{Hom}_G(\varphi, \varphi)$. φ irreducible

Schur: $T' = \lambda I$. Take traces.

$$\Downarrow$$

$$\underbrace{\frac{1}{|G|} \sum_{g \in G} \mathrm{Tr}(\varphi_g T \varphi_g^{-1})}_{= \mathrm{Tr}(T)} = \underbrace{\lambda \mathrm{Tr}(I)}_{= \lambda n}$$

Assume $\varphi: G \to GL_n(\mathbb{C})$ irreducible.

$T \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \longleftrightarrow X = (x_{ij}) \in \mathrm{Mat}_{n \times n}(\mathbb{C})$

$\Downarrow$ $\qquad\qquad\qquad\qquad \Downarrow$

$T' = \lambda I \qquad\qquad\qquad\qquad X' = \lambda I$

so

$$X' = \frac{1}{|G|} \sum_{g \in G} \underline{\varphi_g} \, \underline{X} \, \underline{\varphi_{g^{-1}}} \qquad X = (x_{ij})$$

$(x'_{ij}) =$

matrix entries

$$x'_{ij} = \frac{1}{|G|} \sum_{g \in G} \sum_{u,v} \varphi_{iu}(g)\, x_{uv}\, \varphi_{vj}(g^{-1})$$

$$x'_{ij} = \sum_{u,v} \left[\frac{1}{|G|} \sum_{g \in G} \varphi_{iu}(g)\, \varphi_{vj}(g^{-1}) \right] x_{uv}$$

$$X' = \lambda I, \qquad \lambda = \frac{1}{n} \mathrm{Tr}(X)$$

$$\lambda = \frac{1}{n} \sum_{u,v} x_{uv} \delta_{uv}$$

$$x'_{ij} = \sum_{u,v} \left[\frac{1}{n} \delta_{uv} \delta_{ij} \right] x_{uv}$$

so

$1 \le u, v \le n$ $\quad$ fixed $1 \le i, j \le n$

$$0 = \sum_{u,v} \left[\frac{1}{n} \delta_{uv} \delta_{ij} - \frac{1}{|G|} \sum_{g \in G}' \varphi_{iu}(g) \varphi_{vj}(g^{-1}) \right] x_{uv}$$

$X = (x_{uv})$ is arbitrary.

plug in $x_{uv} = 1$, other entries $x_{uv} = 0$ $\Longrightarrow$

$$\boxed{\frac{1}{n} \delta_{uv} \delta_{ij} = \frac{1}{|G|} \sum_{g \in G}' \varphi_{iu}(g) \varphi_{vj}(g^{-1})}$$

$1 \le i, j, u, v \le n$

Special case: $i = u$, $j = v$

$$\boxed{\frac{1}{n} \delta_{ij} = \frac{1}{|G|} \sum_{g \in G}' \varphi_{ii}(g) \varphi_{jj}(g^{-1})}$$

$$\langle \chi, \chi \rangle = \sum_{i,j} \frac{1}{|G|} \sum_{g \in G}' \varphi_{ii}(g) \varphi_{jj}(g^{-1})$$

$$= \sum_{i,j} \frac{1}{n} \delta_{ij} = \frac{1}{n} \cdot n = 1 \quad \checkmark$$